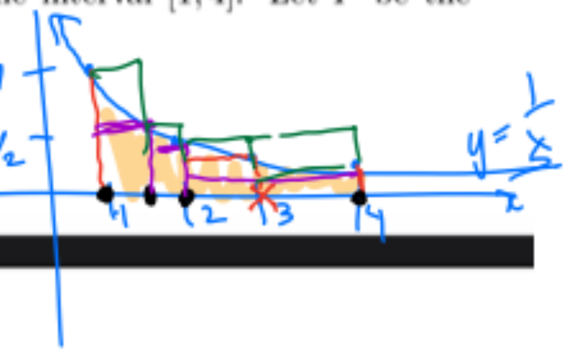
Exercise 7.2.2. Consider $f(x) = 1/x$ over the interval $[1,4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

L U



(a) Compute $L(f,P)$, $U(f,P)$, and $U(f,P) - L(f,P)$.

(b) What happens to the value of $U(f,P) - L(f,P)$ when we add the point 3 to the partition?

(c) Find a partition P' of $[1,4]$ for which $U(f,P') - L(f,P') < 2/5$.

(a) $f(x) = \frac{1}{x}$ is a decreasing fcn on $(0,\infty)$, because

if $0 < x < y$, then $\frac{1}{x} > \frac{1}{y} \Rightarrow f(x) > f(y)$.

Thus for any partition, if $x_j < x_{j+1}$

$$m_j = \inf\{f(x) : x_j \le x \le x_{j+1}\} = f(x_{j+1}) = \frac{1}{x_{j+1}}$$

$$M_j = \sup\{f(x) : x_j \le x \le x_{j+1}\} = f(x_j) = \frac{1}{x_j}$$

$$P = \{\overset{x_0}{1}, \overset{x_1}{\tfrac{3}{2}}, \overset{x_2}{2}, \overset{x_3}{4}\}$$

$$L(f,P) = m_1\left(\tfrac{1}{2}\right) + m_2\left(\tfrac{1}{2}\right) + m_3(2)$$

$$= \tfrac{2}{3}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2}\right) + \tfrac{1}{4}(2)$$

$\frac{1}{3/2}$

$$= \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{2} = \frac{4+3+6}{12} = \boxed{\frac{13}{12}}$$

$$U(f,P) = M_1\left(\tfrac{1}{2}\right) + M_2\left(\tfrac{1}{2}\right) + M_3(2)$$

$$= (1)\left(\tfrac{1}{2}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) + \tfrac{1}{2}(2)$$

$$= \tfrac{1}{2} + \tfrac{1}{3} + 1 = \frac{3+2+6}{6} = \boxed{\frac{11}{6}}$$

Thus, $U(f,P) - L(f,P) = \frac{11}{6} - \frac{13}{12} = \frac{22}{12} - \frac{13}{12}$

$$= \frac{9}{12} = \boxed{\frac{3}{4}}$$

(b) $P' = P \cup \{3\}$

$$= \{\underset{x_0}{1}, \underset{x_1}{\tfrac{3}{2}}, \underset{x_2}{2}, \underset{x_3}{3}, \underset{x_4}{4}\}$$

$\Rightarrow U(f,P') - L(f,P')$

$$= \sum (M_k - m_k)\,\Delta x_k$$

$$= (M_1 - m_1)\tfrac{1}{2} + (M_2 - m_2)\tfrac{1}{2} + (M_3 - m_3)(1)$$
$$+ (M_4 - m_4)(1)$$
$$= \left(1 - \tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{2}{3} - \tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right)(1)$$
$$+ \left(\tfrac{1}{3} - \tfrac{1}{4}\right)(1)$$
$$= \tfrac{1}{2} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} = \boxed{\tfrac{1}{2}}$$

Related Question: Prove $\frac{1}{x}$ is Riem. integrable on $[1,4]$.

Scratch Work:

I would need $U(f,P) - L(f,P) < \varepsilon$

Let's try even partitions. $x_k = 1 + \frac{k}{n}$

$$n \in \mathbb{N} \quad P_n = \{\underset{x_0}{1},\ \underset{x_1}{1+\tfrac{1}{n}},\ 1+\tfrac{2}{n},\ \dots,\ \underbrace{1+\tfrac{3n}{n}}_{=4}\}$$

(3n intervals)

$\Delta x = \frac{1}{n}$

$$U(f,P_n) - L(f,P_n)$$

$$= \sum_{k=1}^{3n} (M_k - m_k)\left(\frac{1}{n}\right)$$

$$M_k = \sup\left\{ \frac{1}{x} : 1 + \frac{k-1}{n} \le x \le 1 + \frac{k}{n} \right\}$$

$$= \frac{1}{1+\frac{k-1}{n}} = \frac{n}{n+k-1}$$

$$m_k = \inf\left\{ \frac{1}{x} : 1 + \frac{k-1}{n} \le x \le 1 + \frac{k}{n} \right\}$$

$$= \frac{1}{1+\frac{k}{n}} = \frac{n}{n+k}$$

$$(M_k - m_k) = \frac{n}{n+k-1} - \frac{n}{n+k}$$

$$\therefore U(f, P_n) - L(f, P_n)$$

$$= \sum_{k=1}^{3n} \left(\frac{n}{n+k-1} - \frac{n}{n+k} \right) \frac{1}{n}$$

$$= \sum_{k=1}^{3n} \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right)$$

$$= \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}}$$

$$\cdots + \cancel{\frac{1}{n+3n-1}} - \frac{1}{n+3n} = \frac{1}{n} - \frac{1}{n+3n}$$

$$= \frac{1}{n} - \frac{1}{4n} = \frac{4-1}{4n}$$
$$= \frac{3}{4n} < \varepsilon$$

$$\frac{4n}{3} > \frac{1}{\varepsilon}$$
$$n > \frac{3}{4\varepsilon}$$

Actual Proof: $\forall \varepsilon > 0$, choose $n \in \mathbb{N}$ s.t. $n > \frac{3}{4\varepsilon}$ (which we can do by Archim).

Then let $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots, 1+\frac{3n}{n} = 4\}$ be a partition of $[1,4]$.

Then

$$U(f, P_n) - L(f, P_n)$$

$$= \sum_{k=1}^{3n} (M_k - m_k)\,\Delta x_k$$

$$= \sum_{k=1}^{3n} \left(\sup\left\{\frac{1}{x} : 1+\frac{k-1}{n} \le x \le 1+\frac{k}{n}\right\} - \inf\left\{\frac{1}{x} : 1+\frac{k-1}{n} \le x \le 1+\frac{k}{n}\right\}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{k=1}^{3n} \left(\frac{1}{1+\frac{k-1}{n}} - \frac{1}{1+\frac{k}{n}} \right) \frac{1}{n}$$

$$= \sum_{k=1}^{3n} \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right)$$

$$= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}$$

$$= \frac{1}{n} - \frac{1}{4n} = \frac{3}{4n} < \frac{3}{4\left(\frac{3}{4\varepsilon}\right)} = \varepsilon.$$

$\therefore$ $f(x) = \frac{1}{x}$ is Riem int. on $[1,4]$. $\blacksquare$

(c) Find Partition P'' s.t.

$$U(f,P'') - L(f,P'') < \frac{2}{5}.$$

By our calculation, if

$$n > \frac{3}{4\varepsilon} = \frac{3}{4\left(\frac{2}{5}\right)} = \frac{15}{8} = 1\frac{7}{8},$$

so $n=2$ should work!

$$\left\{1, 1+\frac{1}{2}, 1+1, 1+\frac{3}{2}, 1+2, 1+\frac{5}{2}, 1+3=4\right\}$$

$\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$

$\rightarrow U(f, P_n) - L(f, P_n) = \frac{3}{4n} = \frac{3}{8} < \frac{2}{5}$ ✓.

Thm If $f : [a,b] \to \mathbb{R}$ is continuous, then it is Riem integrable on $[a,b]$.

Scratch $P_n = \{\overset{x_0}{a}, \overset{x_1}{a+\Delta x}, a+2\Delta x, \ldots, a+n\Delta x = b\}$

$\Delta x = \frac{b-a}{n}$

$x_j = a + j\Delta x$

Want $U(f, P_n) - L(f, P_n) < \varepsilon$

$\sum_{k=1}^{n} (M_k - m_k)\left(\frac{b-a}{n}\right)$

$M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$

f is cont $\Rightarrow$ f achieves its max & min on $[x_{k-1}, x_k]$.

$= \max\{f(x) : x_{k-1} \le x \le x_k\} =: f(c_1)$

Sim. $m_k = \min\{f(x) : x_{k-1} \le x \le x_k\} = f(c_2)$

Want bound on $|f(c_1) - f(c_2)| <$

Note: f is unif cont. on $[a,b]$
since f is cont & $[a,b]$ is compact.

$\forall \tilde{\varepsilon} > 0$, $\exists \delta > 0$ s.t.

if $|c_1 - c_2| < \delta$ then
$|f(c_1) - f(c_2)| < \tilde{\varepsilon}$.

Choose n
s.t. $\frac{b-a}{n} < \delta \Rightarrow b-a < \delta n \Rightarrow$

$$n > \frac{b-a}{\delta} \quad \checkmark$$

$$U(f,P_n) - L(f,P_n)$$

$$= \sum_{k=1}^{n} (M_k - m_k)\left(\frac{b-a}{n}\right)$$

$$< \sum_{k=1}^{n} \tilde{\varepsilon}\left(\frac{b-a}{n}\right) = \tilde{\varepsilon}(b-a).$$

We will need $\varepsilon = \tilde{\varepsilon}(b-a)$

$$\tilde{\varepsilon} = \frac{\varepsilon}{b-a}$$

Actual proof of thm: With given,

$\forall \varepsilon > 0$, let $\tilde{\varepsilon} = \frac{\varepsilon}{b-a} > 0$.

Since f is continuous on the compact set $[a,b]$, it is uniformly continuous on $[a,b]$. Thus, $\exists \delta > 0$ s.t. if $|x-c| < \delta$ and $x, c \in [a,b]$, then $|f(x) - f(c)| < \tilde{\varepsilon}$.

Next, by Archie, $\exists n \in \mathbb{N}$ s.t. $n > \frac{b-a}{\delta}$.

Choose $P_n = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, \underbrace{a + \frac{n(b-a)}{n}}_{b}\}$.

Then $U(f, P_n) - L(f, P_n)$

$$= \sum_{k=1}^{n} (M_k - m_k) \frac{b-a}{n},$$

where $M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$,

where $x_j = a + j\left(\frac{b-a}{n}\right)$.

Since f is cont on $[x_{k-1}, x_k]$, by EVT, f achieves its sup at $x = c_k \in [x_{k-1}, x_k]$ and its inf at $x = d_k \in [x_{k-1}, x_k]$.

$$\Rightarrow M_k = f(c_k) = \max\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$m_k = f(d_k) = \min \quad '' \quad .$$

Then, since $c_k, d_k \in [x_{k-1}, x_k]$,

$$|c_k - d_k| < \frac{b-a}{n} < \frac{b-a}{\left(\frac{b-a}{\delta}\right)} = \delta$$

$$\Rightarrow |f(c_k) - f(d_k)|$$

$$= f(c_k) - f(d_k) < \tilde{\varepsilon} = \frac{\varepsilon}{b-a}.$$

Thus, $U(f, P_n) - L(f, P_n)$

$$\leq \sum_{k=1}^{n} |f(c_k) - f(d_k)| \frac{b-a}{n}$$

$$< \sum_{k=1}^{n} \left(\frac{\varepsilon}{b-a}\right)\left(\frac{b-a}{n}\right) = \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon.$$

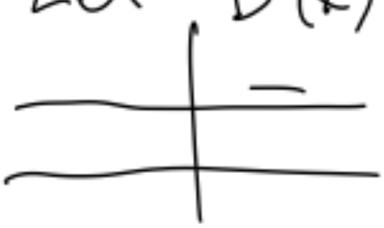
$\therefore$ f is Riem. integrable on $[a,b]$. $\blacksquare$

Remark: One can also show that if f is piecewise continuous on $[a,b]$ (i.e has a finite # of discontinuities), then f is Riem integrable on $[a,b]$.

[to prove: first show that if $x_0 \in [a,b]$, and $f|_{[a,x_0)}$ and $f|_{(x_0,b]}$ are Riemann integrable, then f is Riem. int. Then apply the result to a function with one discontinuity, then two, then 3, …].

Example of a Weird fcn that is not Riemann integrable:

Let $D(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^c \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$.



On $[a,b] \subseteq \mathbb{R}$, $a < b$.

$P = \{a = x_0, x_1, \dots, x_n = b\}$

$U(D, P) - L(D, P)$

$= \sum_{k=1}^{n} (M_k - m_k)\underbrace{(x_k - x_{k-1})}_{\Delta x_k}$,

where $M_k = \sup\{D(x) : x_{k-1} \leq x \leq x_k\}$

$m_k = \inf\{D(x) : x_{k-1} \leq x \leq x_k\}$

$\Rightarrow M_k = 1$

$m_k = 0$

$= \sum_{k=1}^{n} (1 - 0)(x_k - x_{k-1})$

$= \sum_{k=1}^{n} (x_k - x_{k-1}) = x_1 - x_0 + x_2 - x_1$

$+ \dots \ x_n - x_{n-1}$

$= x_n - x_0 = b - a.$

Can't make it $< \varepsilon$.

D is not Riem. integrable on any $[a,b]$.